Feasible Real-time Nonlinear Model Predictive Control

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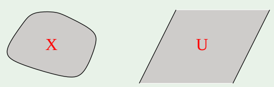
1 Nonlinear Plant Model

Discrete Time:

$$x_k = F(x_{k-1}, u_{k-1})$$
$$y_k = g(x_k)$$

with the constraints

$$x_k \epsilon \mathbf{X} \qquad \text{and} \qquad u_k \epsilon \mathbf{U}$$



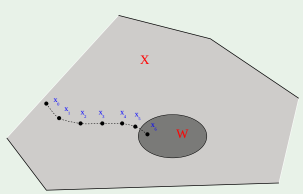
2 Suboptimal MPC

Challenges:

- Computation of a global optimum for the cost function is not always fast or reliable.
- The endpoint stability constraint complicates the optimization problem.
- Poor initial guesses may prevent optimization algorithm from making progress.

Proposed Solution:

- Calculate the region around the origin that can be stabilized by a linear control law and require the final state to be in this region, **W**.
- Rather than finding the optimal sequence of control moves, we settle for a feasible sequence of moves that decrease the cost function from its prior value.
- Most of the computation to determine the endpoint region can be performed off-line to increase speed of performance.



MPC with terminal region:

$$\min_{u^N} \Phi_k = \sum_{j=0}^{N-1} (x_{k+j}^T Q x_{k+j} + u_{k+j}^T R u_{k+j} + \Delta u_{k+j}^T S \Delta u_{k+j}) + x_{k+N}^T P x_{k+N}$$

with the additional constraint

$$x_{k+N} \in W_\alpha$$

3 Sequential Quadratic Programming

- We optimize using a sequential quadratic programming (SQP) technique, which has the following properties:
  - ∗ Solves a series of quadratic programs (QPs) which approximate nonlinear problem at the current iterate.
  - ∗ Each QP has a quadratic objective function and linear constraints.
- We focus on feasibility of each quadratic program. This requirement will allow us to run the controller suboptimally, if desired.
- We solve each QP with a specially structured solver, which exhibits linear growth with horizon length. Normal solvers grow cubically.

The structure of the KKT conditions to be solved is:

$$\begin{bmatrix}
R_0 & B_0^T & & & & & & & & & & & & & \\
B_0 & 0 & -I & & & & & & & & & & & & \\
 & -I & Q_1 & M_1 & A_1^T & & & & & & & & & & \\
 & & M_1^T & R_1 & B_1^T & & & & & & & & & & \\
 & & A_1 & B_1 & 0 & -I & & & & & & & & & \\
 & & & & -I & Q_2 & M_2 & A_2^T & & & & & & & \\
 & & & & & & & \ddots & & & & & & & \\
 & & & & & & & & M_{N-2}^T & R_{N-2} & B_{N-2}^T & & & & \\
 & & & & & & & & A_{N-2} & B_{N-2} & 0 & -I & & & \\
 & & & & & & & & & & -I & Q_{N-1} & M_{N-1} & A_{N-1}^T & \\
 & & & & & & & & & & & M_{N-1}^T & R_{N-1} & B_{N-1}^T & \\
 & & & & & & & & & & & A_{N-1} & B_{N-1} & 0 & -I \\
 & & & & & & & & & & & & & -I & \Pi
\end{bmatrix}
\begin{bmatrix} u_0 \\ \lambda_1 \\ x_1 \\ u_1 \\ \lambda_2 \\ x_2 \\ \vdots \\ u_{N-2} \\ \lambda_{N-1} \\ x_{N-1} \\ u_{N-1} \\ \lambda_N \\ x_N \end{bmatrix}
=
\begin{bmatrix} r_0^u \\ r_0^p \\ r_1^x \\ r_1^u \\ r_1^p \\ r_2^x \\ \vdots \\ r_{N-2}^u \\ r_{N-2}^p \\ r_{N-1}^x \\ r_{N-1}^u \\ r_{N-1}^p \\ r_N^x \end{bmatrix}$$

This system of equations is reduced to a time-varying discrete-time Riccati equation and solved.

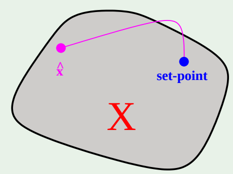
4 Terminal Constraint

We exclude the terminal region constraint in the quadratic program. This is done for a number of reasons:

- If the horizon length N is chosen to be too short, the problem is infeasible.
- If the horizon length is barely long enough to reach the terminal region, the closed-loop solution may differ appreciably from the open-loop prediction.
- Ellipsoidal constraints in quadratic programs are not exact; the constraint would be approximated.

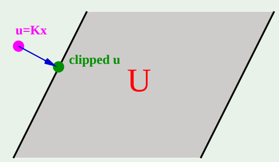
5 Soft Constraints

We treat the state constraints $x \in X$ as soft constraints. The optimizer is allowed to violate these constraints, but penalizes such violations.



6 Feasibility

Generate feasible sequence using feedback law $u_k = Kx_k$. If $u_k \notin U$, then we clip u_k:



Since we have soft state constraints, we have a feasible initial guess. Using this initial guess in an SQP method results in a feasible sequence with no worse than the same value of the objective.

7 NMPC vs. linear MPC

We will study the exothermic reaction $A \to B$ in an externally cooled CSTR.

First, we attempt to reach a set-point of $C_A = 0.21M$, $T = 369.7K$ from an initial point of $C_A = 0.4M$, $T = 300K$:

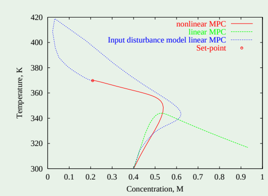


Figure 1: Comparison of NMPC to linear MPC

8 Disturbance Rejection

Now we try to maintain the set-point $T = 350K$:

Figure 2: Output random disturbance

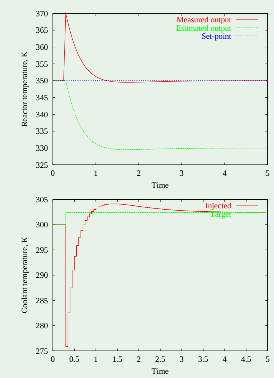


Figure 3: Output step disturbance

9 Future Work

- Structured moving horizon estimation to incorporate constraints in state estimator
- Incorporation of second derivative information to speed convergence
- Practical application without terminal region